Marcelo Vásconez
http://equaguia.com/mv

Dissertation Chapter 1: INTRODUCTION

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0.- Purposes and Importance of the Research

The present work can perhaps be classified as belonging to the field of philosophy of logic, since one of the problems it will examine is 1) what logic is the most adequate to represent and account for the phenomenon of fuzziness, as it appears in reality, our language and thought. Furthermore, this first problem does not come alone, but brings with it a second one, namely, 2) how best to cope with the philosophical aspects involved in the sorites paradox.

Concerning these two difficulties, my intentions are double. The first goal is negative, or destructive, in the sense that my aim is to criticize the standard, bivalent and truth-functional logic (CL, from now on) for being inadequate and deficient in solving both questions. But beside that, there lies the positive end of showing that the approach proposed to replace classical logic, a special blend of many-valued and paraconsistent logics, is in a better position than any other alternative to deal with the problems of fuzziness and the sorites. Indeed, the solution to at least one version of the paradox will consist in declaring invalid the form of the argument, to wit, disjunctive syllogism. Apparently this is a radical move. However, it is important to make it clear from the start that the reform of CL here advocated consists of a demand for an extension[1] of the scope of its jurisdiction, rather than a reduction or curtailment of its power.

Thus the two topics to be investigated are taken to be the main motivations to go beyond CL, towards degrees of truth and contradictions. But bear in mind that, since the whole of CL is incorporated into the new system, there is nothing to lament. Quite on the contrary, it constitutes a necessary enrichment of the received logic.

Therefore, we are going to walk through a path leading to a change of logic. Of course, this has deep consequences. Given that logic sets the limits of what is rational, this notion itself must also be expanded. Indeed, the scope of transformation includes the realm of thinking, for logic puts boundaries to whatever is imaginable or thinkable. Depending on what sort of logic we spouse, the range of thought is going to be narrowed or widened.

The present work aims at describing one rationale behind a particular trend of many-valued and paraconsistent logics.

1.- Preliminaries

Before beginning, some points require our previous attention. First, I present a sketch of the logic undergoing the research, and an examination of different versions of the principles of bivalence, excluded middle and non contradiction. Later I expose some reasons why I prefer to use the word 'fuzziness' instead of the more common 'vagueness'.

a.- Overview of the Logical System to Be Used

Section 2 of "Contradictorial Gradualism vs. Discontinuism" goes here.

b.- The Principles of Excluded Middle and of Non Contradiction

Once several functors of affirmation and negation are in place, we can discern various versions of the traditional principles of excluded middle (PEM, for short), and of non contradiction (PNC). The next table shows the most important schemes:

[1] Indeed, the systems here used, *Aj* and *Aq*, are *strict extensions* of CL, i.e., they keep every tautology and theorem as well as every rule of inference of CL provided that its negation sign be read as a strong negation. In other words, *Aj* and *Aq* are conservative extensions of CL. But of course, there are much more truths and rules to be added.

| | PEM | PNC |
|---|---|---|
| Simple, or Weak | (1) p∨~p<br>(2) Lp∨¬p | (6) ~(p∧~p)<br>(7) ¬(Lp∧¬p) |
| Strong | (3) p∨¬p | (8) ¬(p∧¬p) |
| Absolute | (4) H(p∨~p)<br>(5) Hp∨¬p | (9) H~(p∧~p)<br>(10) ¬(p∧~p) |

(1) and (6), in contradistinction to (3) and (8), are called weak and strong, respectively, due to the kind of negation involved. But in another sense, (2) and (7) deserve to be named 'weak' for they are the least controversial. In fact, (2) affirms that "p" is true to some extent, or else it is totally false, while (7) asserts that it absolutely cannot be the case that "p" is more or less true as long as it is entirely false. Version (8) could also belong to this category of weak principles, because it states that any super-contradiction is completely false, which is something obvious. All these six principles are true.

To the contrary, all four absolute versions are plainly false. (1) and (6) differ from (4) and (9) in that the latter result from prefixing the over-affirmation functor to the former. Thus (4) says that the simple PEM is totally true, whereas (9) says the same thing with respect to the simple PNC. (5) is deduced from (4) by first distributing 'H' over the disjunction, since the functor 'H' is truth-functional, and then by applying the replacement of equivalents, for "H~p" amounts to "¬p": that "it is entirely true that not p" is exactly the same as that "p is completely false". For the same reason, (10) follows from (9).

Notice, finally, the contrast between (6) and (10): the former holds that a simple contradiction is false, but the latter contends that it is totally false. Only (10) excludes contradictions altogether, but not (6), which is compatible with the existence of simple inconsistencies. We can have both at the same time: p∧~p and ~(p∧~p). This is just a contradiction of a second order.

To check the different valuations taken by the mentioned formulations, let me display the truth tables for the PEM, in a penta-valent logic.

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 (1)              (2)             (3)             (4)             (5)
  p  ∨  ~p        Lp  ∨  ¬p       p  ∨  ¬p        H(p∨~p)         Hp  ∨  ¬p
+1   1  0         1   1  0        1  1  0         1   1           1   1  0
±¾   ¾  ¼         1   1  0        ¾  ¾  0         0   ¾           0   0  0
±½   ½  ½         1   1  0        ½  ½  0         0   ½           0   0  0
±¼   ¾  ¾         1   1  0        ¼  ¼  0         0   ¾           0   0  0
-0   1  1         0   1  1        0  1  1         1   1           0   1  1
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We can look at how true each version is. (2) is absolutely true; it is the less contentious. On the other hand, (1) is at least 50% true, or at most 50% false, whereas (3) can diminish below ½ its degree of truth, which can go as low as the lowest designated degree of "p". In flat opposition, versions (4) and (5) are not tautologies[2] at all. And we can see that their pattern is similar: they both are completely false whenever "p" takes an intermediate value. We can call (5) the *Principle of Exclusion of Intermediate Situations* (PEIS). We can see that fuzzy sentences, those which are neither 1 nor 0, imply the complete falsehood of the absolute versions of the PEM. This means that the PEM cannot be a totally true principle.

Parallel semantical considerations apply to the PNC. (7) and (8) are totally true. (6) is at least 50% true, or never more than 50% false. But (9) and (10) are wholly false, whenever there are true contradictions.

[2] A tautology is any formula that only takes designated values independently of the truth values of its component subformulas.

c.- The Principle of Bivalence (PB)

(PB) can be understood in two senses. Strictly speaking, it affirms that there are exactly two truth values: the Truth, and the Falsehood -symbolized as 'T' or '1', and 'F' or '0', respectively- which are jointly exhaustive, and mutually exclusive. By the first requirement it is meant that a sentence must have at least one of them; and by the second, that it has at most one of them. Thus, "p" is either T or F, but not both. In this strong sense, (PB) cannot be respected by any paraconsistent or many-valued logic.

There is another absolute version of the (PB), which is deduced from the absolute version of the PEM, the formula number (5) of the previous section: either "p" is totally true or it is completely false. But the sentence "p" is totally true, or totally false when the truth values '1' and '0', respectively, are assigned to it. Therefore, what the (PB) says in this strict formulation is that either /p/ = 1 or /p/ = 0. Expressed with the help of the 'definitely' operator[3], 'Δ', this strong version of the (PB) says that either "p" is definitely true or it is definitely false.

On the other hand, (PB) may be taken in a loose sense, without the requisite of mutual exclusivity, i.e., as the mere demand that the set of truth values be divided in two exhaustive sub-sets: the designated or true values, and the antidesignated or false values. In this wide sense, few many-valued and paraconsistent systems can still keep the (PB), despite their allowing non classical truth values. "p" is true or false, with lower case 't' and 'f'.

d.- 'Fuzzy', but not 'Vague'

Section 1 of "Contradictorial Gradualism vs. Discontinuism" goes here.

2.- What Are the Problems?

We are going to canvass opposing approaches to two closely related issues: fuzziness and the sorites paradox. We want to know which theory among the alternatives is the most appropriate, or the least problematic. But before delving into the various proposals, we need to have a first look at each topic to be discussed.

3.- The Problems of Fuzziness

a.- The Bearer of Fuzziness

A first point of marked disagreement concerns the subject or bearer of fuzziness. In the next section we will examine in detail what fuzziness is. Here, for present purposes, let us understand fuzziness as the opposite of precision and exactness. When it is denied that the reality is fuzzy, usually what is meant is that objects and properties are sharply bounded, that they have crisp borders, finely carved, not blurry, coarse, or badly defined. On the other hand, a precise word has definite and neat conditions of application.

Once we have a rough and ready initial characterization, we ask what fuzziness is attributed to. Two different perspectives can be distinguished. First, the ontological view, which holds that fuzziness is primarily a feature of items figuring in an ontological inventory of reality, such as objects, properties, relations and facts; they all can be fuzzy. If this is so, then the world itself is fuzzy inasmuch as it contains at least one fuzzy entity. Language also is fuzzy but derivatively. Second, the semantical view, which maintains that fuzziness is a quality only of linguistic expressions, specifically of a sentence or, alternatively, of its components, like nouns and predicates. What is fuzzy is not reality itself, but our description of it by means of language. This stand could be extended to include our mental representation in thought: concepts or judgements

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[3] This functor 'definitely' has been interpreted in several ways. 'Δp' has been taken to mean either that 'p is true in every precisification' by supervaluationists, or as meaning that 'p is clearly true' by agnosticists, and finally in the sense that 'p is completely true' by many-valued logicians. We assume the latter reading, i.e., 'Δp' will be understood as 'Hp'.

can also be fuzzy. Since both parties agree that language is fuzzy, the debate turns around the issue of whether fuzziness is a real characteristic of some kind of entity. So we are confronted with a choice: must fuzziness be taken as something objective, as an intrinsic constituent of the nature of certain things existing in the universe, or should it be reduced to something subjective, somehow dependent on our language, on our conceptual scheme, or on limitations of our perceptive capacities?

If, for a moment, we restrict the discussion to the case of monadic predicates and properties, then the question is: are there fuzzy properties? And this is an ontological query. Nihilists, agnosticists, supervaluationists, pragmatists answer no, while fuzzy and many-valued logicians say yes. We could express the problem in semantical terms: what is the meaning of a fuzzy predicate 'φ'? What does it refer to? And the alternatives are: we can assign to 'φ' either a classical set or a fuzzy set.

Of course, the topic is not easy at all. Maybe, at the end, the issue is whether existence or being is fuzzy. And to positively hold that would be extremely revisionary, or excessively revolutionary. Thus one can understand the strong antagonism to real fuzziness. It is astonishing nonetheless that one of the most rigorous thinkers on the subject, Timothy Williamson (2003: §§ 5, 9) resists the following inferences from (1) to (2), and from (1) to (3):

(1) $\sim\Delta Ba \wedge \sim\Delta\sim Ba$

(2) $\exists\varphi\exists x\ (\sim\Delta\varphi x \wedge \sim\Delta\sim\varphi x)$

(3) $\exists p\ (\sim\Delta p \wedge \sim\Delta\sim p)$

Sentence (1) is a case of fuzziness: Alfred is neither definitely bald, nor definitely fails to be bald. From this it would not follow that there is at least one property and at least one entity such that it is fuzzy whether x has φ[4]. It seems at first sight that the charge of non sequitur is surprising. Sentence (3) affirms that there is at least one fuzzy fact, one which neither definitely obtains nor definitely fails to obtain. It states the thesis that reality is fuzzy. But again (3) would not follow from (1). I do not doubt that a semantics can be devised so that both entailments get frustrated. That is what happens in supervaluationism and agnosticism. The existence of any borderline case does not make reality fuzzy.

Both agnosticists and supervaluationists assign a classic set to a fuzzy predicate. Then, from this perspective, it is natural to maintain that linguistic fuzziness does not have to rely on ontic fuzziness, or that we can perfectly use a fuzzy language in a non fuzzy world (Keefe 2000: 15). However, I just want to record that this is striking. If fuzziness is a trait assigned only to assertions but not to reality, then a mismatch is introduced between the fuzzy language and the exact and precise world. Yet, this consequence already is an inconvenience for a realist who wants to defend a relation between language and the world as direct as possible.

Be that as it may, how can we adjudicate the matter of whether there is real fuzziness in the world? If a fuzzy ontology has any chance of making a compelling case in its favor, then there must be certain phenomena which cannot be satisfactorily explained except by postulating fuzzy sets. What facts are made possible only thanks to fuzzy sets? The reasons to introduce fuzzy sets are the following (Bouchon-Meunier, 1995: 9, 162). Fuzzy sets are required because they allow a progressive passage from a property to its opposite, say, from black to white, avoiding abrupt, sudden transitions, and the imposition of arbitrary cuts. And again, elements may belong to a fuzzy set in a measure less than absolute, so that they are authorized not to belong completely to a set nor to its complement; hence, fuzzy sets permit their members partial membership. For example, a dark grey patch may belong to a high degree to the class of black things and to a low degree to the class of white things. The main idea behind fuzzy sets is that the more an object approaches the typical characterization of a class, the stronger its membership to the class should

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[4] A detailed explanation of the fallacy makes reference to the illegitimate reversal of the scopes of the vagueness operator 'neither definitely... nor definitely...' and the definite description 'the fact that Alfred is bald'.

be. Thus they are in a better position to treat "badly" defined concepts (like 'center of the city', 'old'), or categories that are not well separated and that partially overlap with each other, intermediate situations (almost black), or approximate values (around 2 kilometers).

So there is a range of affairs whose occurrence provide evidence for the hypothesis that there are fuzzy sets, which will be the ontological correlates of fuzzy predicates. We claim that all the mentioned facts are nothing but facets of the existence of graduality in nature. Therefore, the whole issue of whether fuzziness is real or not hinges on whether reality is gradual or not. If the world is gradual, then we need fuzzy sets, and fuzziness will not be merely linguistic or mental. Fuzzy language reflects a fuzzy reality. If there are no degrees of a property, then supervaluationists and agnosticists are correct in believing that properties have sharp limits, and perhaps we will be better off dispensing with fuzzy sets.

If philosophers have avoided to impute fuzziness to the real world, that, I conjecture, is due to a fear of ascribing inconsistency to it. We will see that this misgiving is not grounded.

b.- The Nature of Fuzziness

Likewise, concerning the nature of fuzziness, there is a lack of unanimity as to how to characterize it. It is one of those philosophical concepts about which schools are divided as to which essential features should be included in its definition, each trend having its own preferences. But several attempts have been made to identify the peculiarity of the phenomenon. I review those features which have been proposed as being more prone to be accepted by all sides taking part in the debate: indeterminacy, borderline cases, lack of borders, and sorites susceptibility.

i) Indeterminacy?

A fuzzy fact is constituted by an object which neither possesses nor lacks a property. Think, for example, of a glass which is half filled with water. It is neither full nor empty, supposing one quality is the negation of the other.

(1) $\exists x(\sim\varphi x \wedge\sim \sim\varphi x)$.

If we have to classify a fuzzy object, x, as either φ or not-φ, x defies a neat classification, since it does not properly fall in either category. Applying De Morgan to (1), we get:

(2) $\exists x\sim(\varphi x \vee \sim\varphi x)$

the simple negation of the weak principle of excluded middle. And in this sense, a fuzzy situation falsifies the PEM, and therefore it is a kind of soft indeterminacy. Yet, this is not the same as saying that the simple PEM fails, for it remains true to some degree. Nonetheless, one aspect of fuzziness is that it does entail the negation of the PEM. And vice versa, the ontological principle of excluded middle:

(3) $\forall x(\varphi x \vee \sim\varphi x)$

denies the occurrence of fuzzy entities, since it entails that there cannot be any object x such that it is neither φ nor not-φ:

(4) $\sim\exists x(\sim\varphi x \wedge\sim \sim\varphi x)$.

This is precisely the third possibility that is excluded.

Thus fuzziness and the simple PEM mutually negate each other. We see that (2) denies (3), and (4) denies (1). These two claims of mild incompatibility are correct, as long as we keep the weak negation. Notice, however, that in a paraconsistent system as Aj, we can have the four statements, from (1) to (4), all asserted as partially true, and therefore also false to some extent.

Moreover, when we introduce the operator 'definitely', symbolized by 'Δ', we gain further results. The failure to comfortably cataloging a fuzzy object x happens whenever x is neither definitely φ nor definitely not-φ. Then, translating this into our logical notation, a fuzzy fact is:

(5) $\sim\Delta\varphi x \wedge\sim \Delta\sim\varphi x$.

And, by De Morgan, an instance of fuzziness, as in (5), amounts to:

(6) $\sim(\Delta\varphi x \vee \Delta\sim\varphi x)$.

If we take (5) or (6) by themselves, as statements of an isolated characteristic of fuzziness, they are alright. In fact, rendering (5) in terms of the over-affirmation functor 'H', as in many-valued logics, a fuzzy situation is:

(7) $\sim H\varphi x \wedge \sim H{\sim}\varphi x$
which is the same as
(8) $\sim H\varphi x \wedge \sim \neg\varphi x$
i.e., a fuzzy object neither perfectly possesses a property nor utterly fails to possess it. All these assertions, from (5) to (8), are acceptable. Indeed, (5) or (8) may be seen as grounds for (1).

Nevertheless, if it is claimed that (6) conflicts with the principle of excluded middle, then there is a real problem, because the central idea behind the previous attempt to characterize fuzziness by means of the negation of the simple PEM was that the fuzzy object resisted to be classified into either of two exhaustive alternatives. Notice that this is correctly expressed by (1) or (2), but not by (5) nor by (6). Certainly, none of the latter denies two contradictories; indeed, "$\Delta{\sim}\varphi x$" is not the negation of "$\Delta\varphi x$". In order for (6) to be the denial of the weak PEM, the scope of the second negation in the right disjunct should encompass the definitely functor too and not merely the predicative sentence.

One who affirms that (6) denies the principle of excluded middle is immediately committed to take this principle as:
(9) $\Delta\varphi x \vee \Delta{\sim}\varphi x$
i.e., either x is definitely φ or it is definitely not-φ. But this is an absolutist rendering of the PEM, that must be rejected, if there are fuzzy objects or properties. In a many-valued logic, (9) is understood as:
(10) $H\varphi x \vee H{\sim}\varphi x$
which is equivalent to:
(11) $H\varphi x \vee \neg\varphi x$
i.e., either x is completely φ or it is not-φ at all, which is the predicative counterpart of formula (5) of section 1c above, the Principle of Exclusion of Intermediate Situations. This is an all or nothing dilemma. (9) and (11) tell us that the two mentioned radical alternatives exhaust the range of possibilities. From a semantical point of view, what the PEIS tells us is that a sentence is either totally true or entirely false. And, finally, if we undo the distribution of the affirmation functor 'H' over the disjunction in (10), we obtain:
(12) $H(\varphi x \vee {\sim}\varphi x)$
which affirms that the simple principle of excluded middle is completely true, which is the predicative form of scheme (4) of paragraph 1c.

Yet this is a mistake. If there are fuzzy situations, and we see no reason to deny this, then the simple PEM must be false, up to a point. More accurately, the PEM is false in exactly the same extent as there are fuzzy facts. And vice versa, these are unreal to the degree that the weak PEM is true. We saw this before.

But the incompatibility between (6) and (9) is of another sort, it is downright. Because of the presence of 'Δ', both claims can only receive a classical truth value. And, since one is the direct negation of the other, they are contradictory: there is no way at all that both can be true simultaneously. If (9) is a law of logic, then there cannot be any fuzziness at all in reality. And if (6) correctly represents what a fuzzy fact is, then (9) cannot be a law of logic. Hence we have to decide between them. I keep (6) and reject all the formulations from (9) to (12). Principles like (9) or (11) declare as possible only the extreme cases, excluding all intermediary situations. We would be left with a desert world, containing very few things: those that paradigmatically possess, or fail to possess a given property. All fuzziness would be gone.

Concerning the question of whether the fuzzy object, x, is utterly indeterminate, we answer negatively, since that could only happen if x utterly did not possess any of the opposite properties, φ or not-φ, in any degree whatsoever. But I do not see how this could happen, given the *Principle of Inverse Co-variance of Opposites*:
(ICO) the more an object is φ, the less non-φ it is, and vice versa.
For example, the more the door is open, the less it is closed. There cannot be any strong indeterminacy with respect to φ if we are able to place the purportedly indeterminate object in a series of elements ordered by the relation of 'being more φ than'. In this series, as we move from one extreme to the other, we know that each consecutive element is less φ in the same extent as it is

more not-φ. If there are degrees, which will be established in section ____, then we can escape the false alternative presented by the Principle of Exclusion of Intermediate Situations.

In conclusion, we can say that a fuzzy object is weakly indeterminate, in the sense that it neither is φ nor not-φ, because it is not completely either. Therefore, the simple principle of excluded middle is false, not entirely, but partially, though it is also partially true. Moreover, we have refused to acknowledge the absolute versions of the PEM. It is wholly false that only what is definitely so and so has a place in the ontology. It is absolutely essential to include a third option beside the extremes.

Notwithstanding this first approximation to fuzziness is purely negative: it only tells us what a fuzzy object is not. Fuzziness is a sort of *tertium quid* with respect to the opposites, neither the one nor the other. But exactly what is it? Eliminating certain possibilities may help us to begin understanding what fuzziness is, but something else must be added; a positive characterization is wanted.

ii) Borderline Cases

A second attempt to characterize fuzziness appeals to the notion of a borderline case. Originally, this notion is intended as an explanation of semantic fuzziness: an expression is fuzzy if it has the possibility of borderline cases. For example, x is a borderline case of a predicate 'φ' when x neither is within the extension of φ nor outside of it, but x -as it were- sits exactly on the dividing limit. That is, it is presupposed beforehand that the space is split in two regions, one belonging to the field of application of the expression in question, and the other beyond its range, and that there is a line effecting the partition. Further, the location of the fuzzy case is initially indeterminate, in the sense that it is neither in nor out, but immediately after it is specified that its proper place is precisely on the boundary.

Now, let us try to give the idea an ontological twist. The ontological space of properties is demarcated by frontiers separating each quality from the rest. Imagine the enclosed area of property φ. What is outside the border is not-φ. Hence, an object is fuzzy if it does not lie on either side of the limit but stays right on it.

Have we made some progress with respect to the initial characterization of fuzziness? At first sight, yes. It seems we have advanced one step, since we have assigned a place to the fuzzy object, instead of depriving it of being somewhere on the map. Still, the nature of the borderline case is not wholly elucidated. It remains to be seen whether it continues being neutral or if it belongs to both sides. Is it like staying on the bridge over a river dividing two countries, and therefore in nobody's land, or more like being in the dusk or dawn? If the notion of borderline case is going to be of any help, it is necessary to further explore its nature to have clarity about its status with respect to the two opposites flanking it. Otherwise it seems we have only found a name for the problem we had before.

Is the borderline case indeterminate? It does not seem to be so. Remember our first characterization of fuzzy objects: x neither possesses nor fails to possess property φ.

(1) $\sim\varphi x \wedge \sim \sim\varphi x$

We only need to suppress the double negation in the right conjunct to reach our solution:

(13) $\sim\varphi x \wedge \varphi x$

the borderline case is and is not φ; it possesses both opposites, and therefore is contradictory. An indeterminate situation is an inconsistency in disguise. If dusk is neither day nor night, then it is both, but each only to a certain extent. Dusk is day for there is still some intensity of dim light brightening the sky, although the sun may have just disappeared behind the horizon. And it is night since darkness is almost everywhere, but not quite.

The example of how the day ends and the night begins is a nice illustration of a gradual process. We may say that dusk sets in, if not before, when the lower arc of the solar circumference begins to disappear behind the horizon line. At this moment the luminosity of the sky starts diminishing, though perhaps we do not notice. It may take about half an hour until the upper arc of the sun is no longer visible. At that time the western part of the sky is still light blue although on the eastern part it is dark blue, but nonetheless blue, not black. Meanwhile, the moon and Venus have shown up, and later on the rest of the starts.

If there are contradictory entities, that is so only because the object possesses both opposites in a limited measure. A many-valued logic or a fuzzy set theory can explain why contradictions arise: they are made possible only by degrees in the possession of the opposites.

To try to avoid the passage from (1) to (13) by invalidating double negation will be a highly *ad hoc* move.

If we are disposed to make room for a third category of borderline cases in addition to the extremes, it is better to talk of a borderline zone, for, in the most typical cases, not all objects falling in it will be equally distanced from the poles: some will be closer to, others, farther away from the extremes. So, in general, this region is not homogeneous, but it is internally differentiated. If we consider the segment of the spectrum of colors going from yellow to red, we can discriminate there more than one shade of orange. Some bands will be reddish orange, while others yellowish orange. This example also serves to illustrate the fact that this middle area consists of an overlap of opposites. Orange has as many nuances as there are ways of mixing the proportions of each of its ingredient colors, red and yellow.

Summarizing this second point we can say that most standard cases of fuzziness demand the recognition of a third intermediary thick zone of variegated borderline cases as a welcome and necessary addendum to the logical landscape. Yet the status of the fuzzy object as a borderline case is not (strongly) indeterminate but contradictory.

To end this topic, we should address the question of the limits of this zone of intermediate cases. Does it have precise borders or not? If it itself is imprecisely delimited, then the scope of the borderline cases is fuzzy. Authors talk here of second order vagueness or higher order vagueness. If borderline objects were indefinite, being in a third category other than φ and not-φ, and the set they form were itself indefinite, in the sense that there were some objects which neither belonged nor failed to belong to this third category, then it would be vague which objects are vague.

Personally, I think that, in the case of a bounded property, as explained in the next section, there are clear limits to which objects are fuzzy and which are not. Fuzziness begins the moment we depart from the total possession or lack of a property. And these two conditions are exactly determined. Any object which neither completely has a property φ nor altogether lacks φ qualifies as a fuzzy object. If we had an ordered series of objects such that the first, $a_0$, is completely φ, the last, $a_z$, is utterly not-φ, and the intermediate members diminish their degree of possession of φ so that there is a gradual transition from the first to the last, then except both, $a_0$ and $a_z$, all the rest are fuzzy. So there is no indefinition nor unclarity concerning which entities are fuzzy. The borderline cases are definitely circumscribed.

Nonetheless, there is a sense in which the range of fuzzy objects may be said to be fuzzy. This will be explained in the chapter dealing with the gradualist approach, ________.

iii) No Boundaries?

Let us now review an alleged third mark of fuzziness. It has been claimed that a fuzzy object or property is one which does not have limits. An illustration of this is the property of being tall as applied to human beings: there is no precise minimum height for persons to qualify as being tall. The same lack of cutoff points would affect colors. If we take the spectrum, where exactly does red begin and end? Again, how many hairs does a man need to have in order to be bald? How much money must a person possess to be rich? It seems there is no accurate, natural answer to these questions, even if all the contextual factors are laid down. And, consequently, it is concluded that there are no borders. Other properties and objects satisfying this condition are: young, cold, fast, big, heavy, far, heap, tadpole, chair, etc. So first we are asked where we should draw the line that marks the end of a property and the beginning of its opposite; and the answer is expected to be sharp. But from our apparent impossibility to solve the question, it is deduced that there is no border limiting the property. Notice that the rejection of frontiers is not restricted only to a certain kind of them, but it is general. It is not that fuzzy entities lack precise, clear-cut borders, but possess fuzzy borders; rather, they are boundaryless.

Of the three features that we have seen so far, this is the one which has the least number of supporters. However, first, we must concede that the question concerning where the exact borderline should be drawn is sensible, and that it admits of a variety of reasonable answers. For example, agnosticists declare that there is one exact limit but which is unknowable to us, whereas supervaluationists maintain that there are several precise demarcations possible, all equally legitimate. Furthermore, we also have to admit that the existence of continuity in nature makes it very difficult for us to set limits to fuzzy properties, because a border of a quality divides the entities into those that possess it from those that do not. And it seems that we do not find these borders anywhere. They are not apparent.

Despite these two concessions, we have reservations about the conception of fuzzy entities as those which are boundaryless. One reason for this is that we believe that a fuzzy property has many borders. In order to show this, consider two things, x and y, which are both φ, but x less than y. For example, x is less tall than y, though x is also tall. My claim is that there is a limit whenever there is, between two things, a relation of inferiority in the possession of a property. Why? Because of the already mentioned Principle of Inverse Co-variance of Opposites:

(ICO) x is less φ than y in the same extent as x possesses more of the opposite non-φ than y.

Simias is less tall than Phaedo iff he is shorter than Phaedo. And if Simias is shorter than Phaedo, then he is short, in virtue of the *Aristotelian Rule for Comparisons*:

(ARC) nothing can have a property in a greater or lesser degree if it does not possess it unqualifiedly;

otherwise, how could something have less or more of a property if it does not possess it at all? But to be short is not to be tall. Hence, Simias is not tall. Consequently, to have less of a property is not to have it. Therefore, when we pass from y to x, we pass from φ to non-φ; that is, there is one limit between the two. The property 'tall' has a first frontier here. And this case can be reiterated many times, as many as there are pairs of individuals that can bear the relation of inferiority. So, if Socrates is less tall than Simias, by the (ICO) Principle, he is shorter than Simias, and then, by the (ARC), Socrates is short, and not tall. This means that the property 'tall' has a second limit between Simias and Socrates. And so on. Thus fuzzy expressions are multiply bordered. The existence of degrees entails the existence of boundaries.

But remark that the kind of limit proposed here is not of the old type, which partitions the universe into two mutually exclusive and disjoint classes. It is a paraconsistent border, one which allows things not possessing the property to be in the same class as those possessing it. The situation is a little more complicated than in the traditional picture. Let us return to the previous example. We saw that Simias falls on the negative side of the first limit: he is not tall. However, Simias is taller than Socrates; therefore, he is tall, in view of the (ARC). And because Socrates is short, a second limit occurs between them. Relatively to this second boundary, Simias is tall; he can be grouped together with Phaedo, from whom he was separated by the first limit. The fact that Simias is not tall makes the first boundary what it is: something that cuts off; but, since Simias is also tall, the first boundary does not discriminate: there are tall people on both sides of the fence. This is why the boundaries are not rigid, or crisp, but allow an overlap of opposites. The frontier is relaxed, permitting migration in both directions, and border crossings. The limits divide and do not divide.

We conclude, fuzzy entities do have soft borders.

Two last observations should be made before we leave this issue. First, if a logical system S does not have degrees of truth, as it is the case in CL, it cannot express neither weak denial nor partial affirmation. That is, if the meaning of '~' is given by its truth table, then, in the absence of non standard truth values, one cannot specify the values taken by "~p". And again, in S, there is no way to express that a sentence "p" is more or less true but not completely true (Lp ∧ ~Hp). In a classical framework, if "p" is true, it is totally true. That entails that the notion of border employed by one who lacks gradualist resources, has to be maximalist, to wit: a boundary divides entities that are completely so-and-so from those which are utterly not so-and-so: Hp on one side, ¬p on the other. More generally, the divide would be between what is definitely φ, from what is definitely not-φ: Δp | Δ~p. This is a *sharp boundary* (Wright 1994: 142; Keefe 2000: 28). By contrast, in a gradualist framework, the border is understood simply as the difference between what is φ and what is not-φ: p | ~p.

Second, there is another sense in which properties may be bounded or not. Let me introduce two definitions, which are important because they give rise to different treatments of the sorites.

A property φ is *bounded*, or closed on both sides, iff there is an object, $a_0$, which definitely possesses it, and another, $a_z$, which definitely lacks it; $a_0$ is properly called a paradigm, prototype, or exemplar of φ; $a_z$ may be called the perfect not-φ instance; it is a paradigm of the opposite of φ. An illustration of this first type of property is the quality of being tall, as it applies to the adult human population currently living. Extreme cases of $a_0$ and $a_z$ are, respectively, the tallest and the shortest men in the world.

φ is *unbounded* or open-ended iff it lacks either $a_0$ or $a_z$. This implies that there is an infinite number of objects which either increase their possession of φ, each object successively exemplifying φ in an ever higher degree without reaching 1, or decrease their degree of possession of φ, each object successively exemplifying φ in an ever smaller degree without reaching 0. An example of this second kind of property which lacks $a_0$ is the quality of being far from the Eiffel Tower. As points augment their distance from the tower, they raise their degree of possession of the property of being far from the tower. But there is no point which is 100% far from the it, since there are always more points which can still be farther away.

iv) Sorites Susceptibility

A fourth and last trait of fuzziness which is likely to receive a majority approval, though not unanimity, is that a fuzzy situation leads to the sorites paradox, i.e., it is responsible for the generation of absurdities. This fourth aspect of fuzziness is dynamic, having to do with small changes in the circumstances. For example, consider the temporal stages of Mary, the unit of measure being a beat of the heart, and compare any two contiguous phases. Then, if we are going to judge Mary as young or not, it is obvious that she will deserve the same treatment at both adjacent stages, in view of the almost perfect identity between both phases. Said otherwise, it does not happen that she is young at stage *i*, but not so at the next stage, *i+1*. This is the major premise of the argument. If we grant this much, and begin with Mary at the age of 12 years old, when she is really young, then, by repeated application of the major premise, we are going to extend the attribution of the quality young throughout Mary's life until she is very old. But, by hypothesis, she is not young at all when she is -say- 95 years old. However, by force of the argument, we conclude that, even at that advanced age, she is still young, which is absurd. Fuzziness thus is charged with being a source of incoherence.

We certainly think it is an essential feature of fuzziness its being the cause of the sorites paradox. What we dispute is that it is incoherent, in the sense that it is absurd. But this is already the topic of paragraph 4. So we abstain from making further commentaries until then.

Recapitulating the four points made so far, we recall that fuzziness has been characterized as being neither definitely φ nor definitely not-φ; that therefore the introduction of a borderline zone is mandatory; that it has many bounds; and that it is what gives rise to the sorites.

c.- The Alethic Status of Fuzzy Sentences

The previous discussion has focused on fuzziness of a fact: it is fuzzy whether an object possesses a property. But we can approach the issue of fuzziness from a semantical point of view, considering what the truth value of a fuzzy sentence is. For example, let 'p' be the sentence 'The cup of coffee is cold', when the coffee is lukewarm. On this respect, there are several ways of looking at the question.

To begin with, classical logic, by its embracing the Principle of Bivalence, dictates that 'p' is either T or F, but not both. Thus indeterminist or contradictorial interpretations of fuzzy sentences are excluded. Remark that (PB) does not necessitate that one know which truth value 'p' has. As a matter of fact, 'p' would be either 1 or 0, but it would be impossible for us to discover which is the case. This is the position of agnosticist philosophers, like Roy Sorensen and Timothy Williamson.

Other authors of an indeterminist conviction, supervaluationists among them, believe that 'p' is neither true nor false, lacking any alethic status whatsoever.

For many-valued logics, 'p' is neither 1 nor 0, but has a non classical truth value, though there is a differentiation among partisans of these logics concerning the designation of the new truth values. When a fuzzy sentence takes a third value, ½, most authors interpret it as indeterminate, neither designated nor antidesignated, whereas some proponents of fuzzy logics supporting the allegiance to the Principle of Bivalence loosely construed permit the positive designation of ½, as well as the introduction of infinite degrees of truth, all of them designated.

And lastly, paraconsistent positions uphold the inconsistent status of 'p': it is both true and false.

So the options concerning the truth value of 'p' are the following: 1) 'p' has exactly one of the classical two truth values; 2) it does not have any truth value; 3) it has a non classical truth value; 4) it has a truth value which is both designated and antidesignated. It is clear that our stand on this particular issue is the third and the fourth.

## 4.- The Sorites Paradox

Let me begin by clarifying the meaning of the two words appearing on the title of this section. First, 'sorites' is an ancient Greek word signifying a heaper, or accumulator, one who adds things. The root from which the word is derived is 'soros', which means a heap, from where the alternative name of the argument comes, namely, 'the heap paradox'. This denomination refers not only to the specific subject matter of the reasoning, but also to its logical structure, since it consists of an accumulation of premises, as we will see immediately.

And second, 'paradox' commonly denotes an argument which has apparently true premises, and is apparently valid[5], but has a contradictory or absurd conclusion. Consequently, if we want to avoid the conclusion, there are basically two ways out (if we discard the possibility of invalidating the argument by the mere presence of fuzzy words in it): either at least one premise is faulty, or the rule of inference used is invalid. There are also philosophers who take the argument as really paradoxical: the premises are certainly true, and the form of the argument is genuinely valid, and yet the conclusion is thoroughly false. That would be for them a proof that fuzziness is irremediably illogical, and that therefore there could not be any logic of fuzziness. This would be outside the realm of logic. We will examine this position in detail later in ______.

### a.- Origins

As for the origin of the paradox, it is only known that its inventor was Eubulides of Megara, a contemporary of Aristotle, in the third century BC. The initial form of the argument consisted in a series of questions: is one many?, is two many?, is three many?, and so on. The context of later discussions in times of the ancient stoicism was a particular aspect of the problem of induction: how many experiences are sufficiently many before we can safely generalize a law in medicine? What is outside our ken is what Eubulides' purposes were. Perhaps the sorites was not merely an intellectual curiosity, a puzzle interesting in itself, but was used as a weapon against a target in a polemic. I conjecture that Eubulides had affinity with the monistic thinking of Parmenides and Zeno of Elea.

### b.- Informal Structure of the Argument

The best known intuitive forms of the sorites are the following two.

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[5] A valid argument, let us recall, is one that, having the premises true to some degree, it is impossible for the conclusion to be absolutely false.

i) The Paradox of the Heap (the sorites *sensu stricto*)

| | |
|---|---|
| First premise: | Zero grains are not a heap. |
| Major premise: | The addition of a single grain cannot make the difference between what is not a heap and what is a heap. |
| Conclusion: | Nothing is a heap. |

ii) The Paradox of the Bald Person (phalakros):

| | |
|---|---|
| First premise: | A hairy person is not bald. |
| Major premise: | The loss of a single hair cannot turn a hairy man bald. |
| Conclusion: | Nobody is bald. |

The structure of both arguments is the same. The first premise consists of an object $a_0$ which paradigmatically, undoubtedly, determinately, possesses the property φ. If we were going to construct a sorites for the color red, then a ripe tomato could serve as our standard of red object. For 'tall', as referring to the height of humans now living, we will have to begin with the tallest man in the world. And so on. If we want to start safely, then the best way is to mention prototypes of φ. Agnosticists will advise us to use unquestionable or evident cases. So this premise is the hardest to question. Yet we will see that even this evident premise has been disputed by nihilists.

The major premise asserts the fact that a minimal difference, scarcely observable, in some underlying dimension, Ψ, on which φ supervenes, does not justify a change in the status of φ concerning its being attributed to, or withheld from some object. The principle behind this will be labeled the Ψ-φ *Correspondence Principle*:

(Ψ-φ C) no tiny alteration in Ψ is capable of creating a significant change in φ

It indirectly asserts the proportionality between variations in Ψ and modifications in φ. Thus, the removal of a grain from a heap is not going to cause its disappearance. Similarly, if I am hairy, the loss of a hair will not make me become bald. In other words, one minute fluctuation in the subvening property Ψ does not drastically affect the membership of *a* to the set of things φ. A very small variation in Ψ cannot cause the transition from φ to not-φ, or vice versa. Thus this major premise also seems obvious in itself. However it has been challenged by many authors who do not have any other option, since they want to leave the logic of the argument intact.

The reasoning may take two forms, according to what kind of process the second premise involves: addition or substraction of units relevant for the application of the predicate, i.e., increase or decrease of the subvening parameter Ψ.

The conclusion is arrived at by continuous reapplication of the process involved in the major premise. If at the beginning, Frank has 100.000 hairs on his scalp, after one hair is lost, he continues being hairy. And again he still is hairy with the loss of one more hair. At each step of alteration in Ψ, there is no radical transformation in φ. His condition of being hairy persists. But not for ever. At the end of the long process, after 100.000 stages, when he loses his last hair, Frank is absolutely bald. Nonetheless, if the reasoning is sound, he must be hairy. And this is really preposterous.

Thus something must be wrong. But what? Both premises clearly seem to be correct. This is the paradox: from seemingly true premises, we arrive at an absurd conclusion. This is our second big problem.

Notice that the conclusions of both the sorites and the falakros generalize the status of the first object to all others. If the prototypical case $a_0$ is φ, then all $a_i$ are φ; and conversely, if $a_0$ is not-φ, then none of the $a_i$ is φ. Anyway, apparently, the effect of the major premise is to deny a transition from one opposite to the other. What it does is to extend the application or non-application of the property φ from the paradigmatic object $a_0$ to all the remaining objects, up to

$a_z$, if there is one. It spreads the fuzzy property, and in this sense we can say that what is fuzzy diffuses itself. Fuzziness is diffusive[6].

c.- Formalization of the Argument

In order to analyze the validity of the argument, it is necessary to give it a logical form, so that the employed rule of inference becomes explicit. The alternatives depend on what formulation the major premise receives. Basically, two different forms will occupy our attention. If the major premise is a conditional, or an implication, then the reiterated rule is *modus ponens* (MP); but if it is a disjunction plus weak negation, then the rule is disjunctive syllogism (DS).

Note that both forms presuppose that φ is bounded: the $a_z$ of the conclusion is not φ at all; otherwise, the argument is not really paradoxical, the conclusion not being absurd, but merely contradictory. Of course, this assumption is not in force if the property φ is open-ended. For a discussion of this last case, see Section e.

i) Conditional Major Premise

| | |
|---|---|
| First Premise: | $a_0$ is φ |
| Major Premise: | If $a_i$ is φ, then $a_{i+1}$ is φ |
| Conclusion: | $a_z$ is φ |
| Rule of inference: | MP: p, p⊃q ⊢ q |

ii) Implicational Major Premise

| | |
|---|---|
| First Premise: | $a_0$ is φ |
| Major Premise: | That $a_i$ is φ implies that $a_{i+1}$ is φ |
| Conclusion: | $a_z$ is φ |
| Rule of inference: | MP: p, p→q ⊢ q |

iii) Disjunctive Major Premise with weak negation

| | |
|---|---|
| First Premise: | $a_0$ is φ |
| Major Premise: | Either $a_i$ is not φ, or $a_{i+1}$ is φ |
| Conclusion: | $a_z$ is φ |
| Rule of inference: | DS: p, ~p∨q ⊢ q |

To be rigorous, a universal quantifier binding $a_i$ and $a_{i+1}$ should be prefixed to all three major premises. Accordingly, we should also have mentioned the rule of universal instantiation, which will allow us to obtain the required particular premises to apply MP and DS. But, since these complications will not affect the subsequent discussion, I have omitted them altogether.

Which form should we prefer? This question is fundamental because we will give the argument a different treatment partly depending on which symbolization we choose for the second premise. The full range of possibilities is open. If the major premises take a conditional or implicational form, then at least one of them is totally false, but the rule is valid. In contrast, if the major premises are disjunctive and use weak negation, then all of them are true, but the rule is invalid. In both cases, the argument is not sound. And, on the other hand, when the property φ is open-ended[7], all premises are true, and the rule is valid, the argument being sound, but the conclusion is not absurd.

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6 In Latin, 'diffusus' is the past participle of 'diffundere', which means to spread out, to scatter.

[7] Remember that, in the present context, a property ϕ is open-ended when there is no last $a_z$ which is a perfect non-ϕ, but there is instead an infinite number of $a_i$ which are ϕ, each in a lesser degree than its predecessor.

In order to support the option for the disjunction plus weak negation reading of the second premise, let us introduce the notion of a soritical series.

d.- The Soritical Series

The Soritical Series underlies the construction of the sorites, and is an ordered collection of elements differing with respect to Ψ. It is this base parameter Ψ which orders the members. In the case of a bounded property, the soritical series has two extremes, each being an ideal instance of the opposites φ and not-φ, respectively; that is, it begins with $a_0$, which is definitely φ, and ends with $a_z$, which is definitely not-φ. In the case of unbounded properties, the soritical series has at most one extreme; at least on one side, there is no end to the series, which will extend *ad infinitum*. Either way, the central feature of the soritical series is that any two contiguous members, $a_i$ and $a_{i+1}$, are subjectively indiscernible concerning their possession of φ, because they barely differ relatively to Ψ. Indeed, due to this very small variation in Ψ, there is a correspondingly tiny, objective dissimilarity in φ among the adjacent members, but it is not observable with the naked eye, it is below our unaided threshold of discrimination, so that, as a matter of fact, we cannot distinguish $a_i$ from $a_{i+1}$. Thus $a_i$ and $a_{i+1}$ are scarcely dissimilar, and therefore, predominantly similar.

It is not easy to avoid a contradictory description of this particular aspect of the soritical series. I believe that the series is indeed contradictory, so a contradiction must appear somewhere sooner or later. But I will not press this issue here.

Let me provide an example of a soritical series. It is a bounded one, for the property 'tall' restricted to our actual human world now. Imagine a sequence of persons ordered by height so that the difference in Ψ between one subject and the next is so little a magnitude that it is imperceptible: one tenth of a millimeter. On one extreme stands the shortest person; on the other, the tallest. Under these conditions, if we are going to compare any two adjacent fellows, we will be entirely unable to detect any difference in tallness among them.

Now, there are at least four ways to capture in logical notation this almost complete similarity with minimal dissimilarity between two elements, $a_i$ and $a_{i+1}$, which are next to each other. As before, the initial universal quantifier is dropped for the sake of simplicity.

| | |
|---|---|
| (SP) | $\varphi a_i \wedge \varphi a_{i+1} \vee. \sim\varphi a_i \wedge \sim\varphi a_{i+1}$ |
| (CP) | $\sim(\varphi a_i \wedge \sim\varphi a_{i+1})$ |
| (Par.P.) | $\sim\varphi a_i \vee \varphi a_{i+1}$ |
| (Pre.P) | $\varphi a_i \supset \varphi a_{i+1}$ |

i) *Similarity Principle* (SP): for any two adjacent members, either both are φ or neither is. This means that two subsequent members should be co-classified. Their likeness grounds the application of the property to both or its withholding from both. Both $a_i$ and $a_{i+1}$ fall on the same side of the boundary. This feature is closely connected with:

*The Fairness Principle*: Like cases must be treated alike.

Fairness is part of what justice means. (SP) is the embodiment of logical equity. The conception of fuzziness based on (SP) could thus adopt the slogan: "vagueness as fairness".

ii) *Continuity Principle* (CP): for any two contiguous elements, it cannot happen that only the former is φ while the latter is not φ. That is, their strong resemblance prohibits any discrimination between them. With other words, (CP) tells us that the border between φ and not-φ is not between $a_i$ and $a_{i+1}$. Again this negative principle can claim to be a facet of justice: it is indeed unfair to treat like cases in an unlike manner. It is felt that a petty difference in Ψ among $a_i$ and $a_{i+1}$ is not enough to give them a contrary treatment.

iii) *Parity Principle* (Par.P): for any two subsequent members, either the first is not φ, or the second is φ.

iv) *Preservation Principle* (Pre.P): for any two flanking neighbors, if one is φ, so is the other.

Observe that the difference between (iii) and (iv) is obliterated in CL, due to the absence of distinct negations. Recall that the conditional used in (Pre.P) is defined by means of strong negation: the truth value of "p⊃q" is equal to that of "¬p∨q". In contrast, the negation employed in (Par.P) is weak. So (iii) and (iv) express quite diverse things.

The first three principles are not independent from each other. (SP) is the stronger of them, in the sense that (CP) and (Par.P) follow from it. These two last principles are logically equivalent, and for that reason, (Par.P) also joins its parent and twin brother in upholding the conception of fuzziness as fairness. The principle which is disconnected from the other three is the fourth one. There is indeed no way to deduce (Pre.P) from them, in view of the contrast between the two negations.

Given that (Pre.P) does not belong to the family, the real alternative is between it and (Par.P). At this moment, we have already all the background needed to answer our previous question: which formulation should we give to the major premise of the sorites? Conditional or disjunctive (with weak negation)? It is time now to make a selection.

Unfortunately for (Pre.P), there is a reason disqualifying it as a suitable representation of the relation among the members of a soritical series: not all of them respect (Pre.P), if the series is bounded. In effect, consider the last two elements of the series, say $a_{z-1}$ and $a_z$. By hypothesis, $a_z$ is not φ at all, for the series is bounded. But $a_{z-1}$ is φ, to some degree however small. (This last claim can be supported as follows. We know that $a_{z-1}$ and $a_z$ are ordered by a relation of inferiority: $a_{z-1}$ is less not-φ than $a_z$. If that is so, $a_{z-1}$ is more φ than $a_z$, by the Principle of Inverse Co-variance of Opposites. And if $a_{z-1}$ is more φ, it is φ, by the Aristotelian Rule for Comparatives). Therefore it results that the conditional '$a_{z-1}$ is φ ⊃ $a_z$ is φ' will be completely false, having a true antecedent but a totally false consequent. That means that the preservation condition is violated at the end of the series.

The same conclusion unfavorable for classical logic would be reached if we replaced the strong negation for the weak one in the (SP), the (CP), or the (Par.P.). That is, the following principles formulated in classical logic become entirely untrue for the last two members of a bounded series.

$(SP)_{CL}$ $\varphi a_i \wedge \varphi a_{i+1} \vee. \neg\varphi a_i \wedge \neg\varphi a_{i+1}$
$(CP)_{CL}$ $\neg(\varphi a_i \wedge \neg\varphi a_{i+1})$
$(Par.P.)_{CL}$ $\neg\varphi a_i \vee \varphi a_{i+1}$

Take, for example, the continuity principle for the strong negation, $\neg(\varphi a_i \wedge \neg\varphi a_{i+1})$. If we suppose that $/\varphi a_i/ = 0.01$, while $/\varphi a_{i+1}/ = 0$, then the second strong negation will be 1, and hence, the conjunction must take the value of the left conjunct, which is 0.01. Therefore the first over-negation shall be 0.

The conclusion we have reached is alarming for CL supporters. We have shown that CL is unable to logically capture the relation among members in a soritical series, since it cannot describe the series by any true principle. This puts CL in a disadvantageous position for it cannot but judge the soritical series as inexistent.

However no such breakdown affects any of the three original principles formulated with weak negation, all of which remain true throughout the series. So we must render the major premise by means of a disjunction and weak negation. Then, the rule used is DS.

e.- Sorites for an Unbounded Property

Beside the sorites using a bounded series, we need to survey another variety which uses an unbounded series, with no $a_z$. We will reserve the name '*slippery slope*' for this kind of reasoning. Let us imagine a collection of balls arranged in a straight line, each one in contact with the next, and let us call the first of them 'A'. The property, or relation, we are interested in is that of 'being close to A'. The hypothesis to be reduced to the absurdum is that there is a ball, Z, on the other extreme of the line, such that it is in no way close to A. We begin our reasoning with ball B, and unconditionally assert that it is close to A, since there is nothing which could be closer to A than what B is. If something is close to A, that is B. Is C close to A? We think it is, since the relation of closeness is taken to be transitive:

B is close to A
C is close to B
-------------------
C is close to A.

The motivation for the transitivity of the closeness relation is that the distancing away from A by a few centimeters cannot bring about the end of the closeness to A. Analogously, not because I move myself one step away from the Eiffel Tower I stop being close to it, or position myself far from it. The same Principle (Ψ-φ C) supporting all major premises of the sorites is operative here: a small difference in Ψ cannot produce a substantial difference in φ. Observe that this first subconclusion does not affirm that C is as close to A as B is. It only says that the relation still holds between A and C, but it does not mention the degree, which, of course, will diminish.

If the validity of the rule is granted, then we apply the same reasoning to the next ball: D is close to C, but C is close to A; therefore, D is also close to A. Again, bear in mind that D will be near to A, but still less than what C is; nonetheless, D is close to A. And we repeat the argument with respect to E, F and all subsequent balls. But this means that, by transitivity, Z too will be close to A. Yet, by hypothesis, Z was absolutely not close to A. Therefore, by *reductio ad absurdum*, there is no such point Z. Hence, every ball is close to A. By the way, the name 'slippery slope' suggests that once you begin to slide, you cannot stop half-way, but have to go all the way down, through an endless path.

In brief, the argument reveals the same logical structure as the others we have seen; in effect, it starts with the concession that the paragon case has the property φ, and demonstrates that all other cases have also φ. The difference is that we do not have an archetype of the opposite of φ, but instead an infinite number of objects each of which exemplifies not-φ in a greater degree than its predecessor but without ever reaching degree 1.

In this formulation of the slippery slope, we have a limited number of attitudes to take to face the reasoning: either we accept the conclusion, or we reject the rule of inference. For a discussion of these alternatives, please see section 6 below.

## 5.- Denials of the Major Premise

Let us see what would happen were we to reject (wholly deny) the truth of at least one major premise of the sorites argument for a bounded property. We have distinguished at least three variants of it, so the negation of the major premise also adopts various formats. Let us enumerate all rejections, on each of which we will comment immediately after. In the first place, informally, the rejection of the major premise would commit us to maintain that the loss of a single hair can turn a hairy man bald, or that the addition of a single grain can make the difference between what is not a heap and what is a heap. In the second place, remember that we identified the Principle (Ψ-φ C) as the one offering the foundation for the major premise. Its refusal would mean that a tiny alteration in Ψ is capable of creating a significant change in φ. And in the third place, formally, if the laws pertaining to the negation of the quantifiers, and the De Morgan principles are in place, then the strong negation of two of the logical versions of the major premise, (CP) and (Par.P), is equivalent to:

(DT) $\exists a_i, a_{i+1}(\mathrm{H}\varphi a_i \wedge \neg\varphi a_{i+1})$

i.e., there is a member in the series such that it is completely φ, whereas its next neighbor is not-φ at all, which we will call the '*Discontinuity Thesis*', and the stand supporting it '*Discontinuism*'. It then affirms that there is a sharp boundary somewhere in the series.

Concerning the Preservation Principle, $\forall a_i, a_{i+1}\ (\varphi a_i \supset \varphi a_{i+1})$, its absolute negation yields a slightly different result, namely, $\exists a_i, a_{i+1}\ (\mathrm{L}\varphi a_i \wedge \neg\varphi a_{i+1})$. What is interesting to note is that the absolute negation of the Similarity Principle, $\forall a_i, a_{i+1}\ (\varphi a_i \wedge \varphi a_{i+1} \vee .\sim\varphi a_i \wedge \sim\varphi a_{i+1})$, produces the existence of a pair of contiguous members, $a_i$, $a_{i+1}$, such that either at least one of them is absurd (completely true and completely false at the same time), or there is a sharp limit between them: while the one is totally φ, the other is altogether not-φ.

Now let us comment on each refusal of the major premise. First, it is incredible that the drop of one hair will turn a hairy man bald. Against this, we must say that it is empirically false;

we observe the contrary. Every time I comb myself, I lose a few hairs, without thereby becoming bald. To truly go bald, I would have to loose thousands of hairs, but not only one.

The root of the mistake is the rejection of the Principle (Ψ-φ C). To believe that there could be a mismatch of proportionality between variations in Ψ and in φ is to assimilate the passage from φ to not-φ to the collapse of an impressive playing-cards castle by the most gentle touch. That again is hard to admit. There should not be a wild discrepancy between changes in Ψ and in φ, because the correspondence between both changes as conveyed by the Principle (Ψ-φ C) seems to faithfully reflect the common sense truisms that the higher a person is, the taller she is, the more money a person has, the richer she is. Perhaps, part of the scruple against (Ψ-φ C) is the existence of seeming counter-examples, like the fact that a few more cents in my purse do not make me richer, for I may be poor. However, this objection does not hold water, because it presupposes that nobody can be rich and poor at the same time. I surmise that once we remove the fear of contradictions, the misgiving against (Ψ-φ C) vanishes thoroughly. Being hosted within a paraconsistent system, the (Ψ-φ C) Principle is immune to attacks of this sort. We find that it seems to be highly implausible that a significant difference in φ can be caused by an insignificant difference in Ψ.

Equally unacceptable is (DT). It postulates the existence of a pair of members in the series such that $a_i$ is the last to be totally φ, while $a_{i+1}$ is the first to be not-φ at all. More exactly, it inserts a sharp divide between the two members. The first problem with this is its arbitrariness. Nothing in nature will justify the exact location of the boundary. Take the property of being an adult human being. What age will mark the beginning of adulthood? Assuredly, whatever exact age you pinpoint, it is not going to have a foundation in reality. Suppose your put the limit at 18 or 25 years old. Whichever the case, the question arises of why not one day before, or one day after. The drawing of the borderline wherever you favor to situate it utterly runs counter to the principle of sufficient reason. There is no ground to put the limit in a particular place rather than somewhere in the vicinity. The boundary can at best receive a practical apology, in virtue of its utility for certain purposes. And thus it appears to be a mere stipulation, without any real basis in the nature of things.

The second reason why a sharp boundary cannot be located between any contiguous members is their intimate likeness. It is because they resemble each other so much that a disparity between $a_i$ and $a_{i+1}$ of the sort postulated by the discontinuist is out of place. The sharp boundary places in antithetical categories two individuals which are indiscriminable. It is like inventing a dissimilitude where all the observational data point to the contrary. When you are backed up by a many-valued and paraconsistent system, you can introduce a dissimilarity where there is a similarity, because you have degrees and tolerate innocuous contradictions. But this is not the case with discontinuism, which imposes upon adjacent members a severance, exaggerating the weight of the divergence. It is true that $a_i$ and $a_{i+1}$ are a little bit unequal, but that is not enough to judge them hardly alike. The repudiation of degrees and contradictions forces on us a spurious dilemma.

Connected with this is the question of fairness. To discriminate between like cases is unjust. Like cases deserve equal treatment, unless there is a relevant reason against that. But a minimal discrepancy among contiguous members is not sufficient to segregate them.

Thus we gather that the rejection of the major premise of the sorites for bounded properties is burden with difficulties: it is empirically false, it goes against common sense truisms, and against the displayed likeness among neighboring members, it is arbitrary and unfair. Therefore, it is advisable to keep the major premises, in accordance with fuzziness. If we do not accept the conclusion of the sorites, we should block the argument by means other than by giving up the premises. That is, we should entertain the strategy of invalidating the rule of inference. Indeed, disjunctive syllogism is invalid for the weak negation, though valid for the strong one. Fortunately, the alteration of CL as contemplated here will only bring gains, with no losses.

We have just argued in favor of maintaining the premises, and against their dismissal. Notwithstanding, this double declaration must be qualified. We do not advocate the total truth of any major premise. That means that they are to some degree false, and this is what partially explains the fact that the sorites is not sound. We know that the paradox intends to prove, by the workings of gradual metamorphoses, the absence of a transition from one property to its opposite. If that is a sophism, we must allow that, by imperceptible transformations, we do go from φ to not-φ. It is to some extent true that:

$$\exists a_i, a_{i+1}(\varphi a_i \wedge \sim\varphi a_{i+1})$$

We accept this for it solely expresses the existence of a soft limit between $a_i$, $a_{i+1}$, due to the weak negation involved.

6.- The Refusal of the Slippery Slope

Let us explore now how we can react to the sorites for unbounded properties, or slippery slope, in the form given above in paragraph 4e. It is clear that to deny its second premise would destroy the first premise too, since both premises hold for exactly the same reason: each ball is close to the preceding one because the distance between them is null; they are side by side. In these circumstances, were we going to refuse the conclusion, we would rather have to deny the transitivity of the closeness relation.

So, let us suppose that, from the facts that B is close to A, and that C is close to B, we cannot infer that C is close to A, insofar as the relation of 'being close to' is not transitive. Unfortunately, this move is fraught with grave consequences, though at first sight it may appear promising. In fact, what follows from the rejection of the transitivity of the closeness relation is that, in order for an object to be close to A, it would need to be in contact with it, that is, completely close to it, as close as B is. Less than that would amount to a failure to being close. Indeed a borderline has been drawn after ball B, so that it is the last one to be close to A, while C is the first to be not close. If the distance from A increases a little, going beyond that between A and B, then that is counted as no longer being close to A. We will call this position '*maximalism*'. In general, it affirms that:

in order for x to be φ, it is necessary that x be absolutely φ.

Thus, only the person who has zero hairs on her scalp is bald, and only Bill Gates is rich. To be cold is to be zero degrees Kelvin. To be good is to be the optimum. To have a property φ is the prerogative of an elite possessing φ at the superlative level. The first privileged members to pass the exacting test will be those included in the McGuiness Book of World Records. Only the best examples will have the right to figure among the instances of each property. And this demand is extended to the property truth also:

*Alethic Maximalism*: a sentence is true only if it is totally true.

From this, the *Maximalization Rule* follows:

$$p \vdash Hp.$$

But this is an extremist position hard to be reconciled with. When the requirements of membership are lifted to the utmost degree, the net result of it is a massive dismissal of "presumed" members, even of those in good standing. Yet that is an unbearable impoverishment of reality. It will deprive us of all non prototypical examples of every property. All imperfect instantiations of φ will not be regarded as genuinely belonging to the set of things φ. Solely the genius would be intelligent. But, since some genius are less than others, just the most brilliant will qualify as intelligent, perhaps one alone.

In fact, this elimination of deficient instances is equivalent to the outright abolishment of fuzziness.

A second unacceptable consequence derived from maximalism is that there will not be degrees of anything different from the maximum. That an attribute comes in degrees means that an object can have it in varying extents, such as large, high, medium, low, slight, or in a greater or lesser extent. If maximalism were true, then nothing could have a property to a certain extent lower than the top degree. Only what is 100% φ would be φ. C could not be less close to A than what B is. There would not be such category as 'less close to A'. C is purely and plainly not close to A, period. The absolute principle of excluded middle would be in force here: or fullness of being or complete non-being. There is no space for intermediate situations.

As a corollary, we would not be able to make comparatives. If there are no degrees, x could not be more or less φ than any y. We could not say that Simias was taller than Socrates, since Simias, being shorter than Phaedo, would not be tall to begin with. If he is not tall in the first place, nor can he be taller. Only the tallest person in the world would be tall (if we consider this property as bounded), and the rest would be not tall. All this is unsatisfactory.

Additionally, other reason preventing the maximalist from being able to make comparatives is that the most direct, simple and illuminating way to understand a comparative, like x is more φ than y, is to analyze it as a comparison of the respective degrees of possession of φ by x and y, that is, the degree in which x is φ is greater than the degree in which y is φ. So, that Rome is warmer than Brussels means that Rome is warm to a greater degree than that in which Brussels is warm. This account is straightforward: the comparative construction 'is φer than' is explained in terms of degrees of φ. x is redder than y iff x is more red than y. But if there are no degrees of any property, then we are deprived of the most obvious manner to clarify the comparatives.

Therefore, in view of the severe difficulties of maximalism, it is commendable to avoid it, and to admit the conclusion of the slippery slope. When a property φ is unbounded, everything is φ.

Yet perhaps there are ways to skirt this conclusion evading maximalism too. A first such an attempt is that maybe the rule of inference, transitivity of closeness, has only *local*, but not global, *validity*, i.e., it is correct provided that solely a few number of applications are made. The problem would be that the rule is reiterated too many times. We should then restrict its use. Thus, we could deduce that C, D, E, F and possibly a few other balls are close to A, but then, at a given point, in order to check the diffusion of the attribute of being close to A, we do not authorize any more instantiations of the rule. Suppose we accept that I is close to A, but even if J is close to I, we block the inference that J is close to A, for transitivity does not have global validity. Thus we would make room for some balls being close to A, and at the same time we would eschew sliding all the way down along the slippery slope.

The problems with this strategy are the same as those of discontinuism, because halting the validity of the rule after some initial applications issues in the introduction of a borderline between balls close to A and those not close to it. But this limit will be arbitrary, will fly in the face of the similarity between neighboring balls, and will be unfair for the contiguous balls flanking the frontier.

Still the corollary of the impossibility of comparatives extracted from maximalism will find resistance. We saw that a natural way to elucidate the comparatives was the following:

x is more φ than y = the degree of x's being φ is greater than the degree of y's being φ.

But there are alternatives to this gradualist conception. One of them is that instead of analyzing the comparative φer in terms of the positive predicate φ, we could take the opposite approach: that of reducing the positive form to the comparative. The meaning of vague predicates would be parasitic on comparatives. First we will have to indicate a minimal threshold for the possession of φ, and then establish that in order for *a* to be φ, it needs to reach at least that threshold. For example, to be tall *a* ought to measure at least 1.8 m., i.e., *a*'s height must be 1.8 m. or more. In general,

'φx' $=_{df}$ the degree of x's Ψ is equal to, or greater than certain standard.

Thus this scheme requires that the vague term be rendered precise, and thus it has the same effect than the tactic of limiting the validity of the rule of transitivity of the closeness relation, and therefore, it also shares the drawbacks of the latter.

We postpone a larger treatment of the topic until section _____.

## 7.- The Tasks of a Theory of Fuzziness

We have had a first contact with the two problems of fuzziness and the sorites paradox, as well as with the main stands we are going to delve into for the rest of the work. Now we bring together the phenomena asking for an explanation, or the most important questions that have to be addressed by any theory of fuzziness.

1.- How is it possible to change from φ to not-φ by means of a soritical series?

2.- What is the nature of the transition from one opposite to the other?
3.- Why does the transition occur?

It is obvious that there is a variety of antagonistic answers, and our purpose is not merely to have a catalogue of possible responses, for we are seeking the truth. We want to know which the best available theory is. No doubt it will have problems. But it is hoped that they are not insurmountable, or that they are less serious than those affecting rival theories. In any case, we need to adjudicate between the contenders. To this end, I propose that the adequate theory is the one offering the best explanation to the problems previously mentioned.

## 8.- Summary

When one approaches the problems of fuzziness and the sorites from the perspective of a many-valued and paraconsistent logic, one is in a privileged position, for, unlike in classical logic, one is able to make distinctions, and therefore, qualified assertions. Indeed, a language based on degrees of truth and the toleration of contradictions has a richer expressibility power. In sharp contrast, classical logic is severely limited in that it cannot properly convey, for example, the relation among members in a soritical series, since not only it cannot distinguish between the Parity and the Preservation principles, but also it cannot formulate a principle that is true of the soritical series.

The main ideas put forward in this introductory chapter have been the following. Fuzziness is acknowledged as being softly, but not entirely, indeterminate. As a result, the simple principle of excluded middle is partially false, though it remains also true, to some degree. But the absolute PEM has to be given up. It has been urged that fuzziness is contradictory, has many mild limits, and generates the sorites paradox, but it is not to blame for the absurd conclusion.

Concerning the sorites, we have expounded two grounds for the plausibility of the major premise: the $\Psi$-$\varphi$ Correspondence Principle, and the Fairness Principle. On the other hand, the rejection of the major premise (discontinuism) and the abandonment of the rule of inference of the slippery slope (maximalism) are fraught with serious problems. What goes wrong with the sorites for a bounded series is the rule of inference: disjunctive syllogism is not valid for the weak negation. And in the case of an unbounded series, we need to accept the conclusion: everything is $\varphi$.

We will have occasion in the following chapters to return to these topics.